9.5 Apply Compositions of Transformations

Before You performed rotations, reflections, or translations.

Now You will perform combinations of two or more transformations.

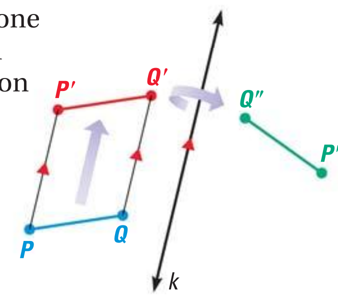
Why? So you can describe the transformations that represent a rowing crew, as in Ex. 30.

Key Vocabulary
- **glide reflection**
- **composition of transformations**

A translation followed by a reflection can be performed one after the other to produce a *glide reflection*. A translation can be called a glide. A **glide reflection** is a transformation in which every point P is mapped to a point P'' by the following steps.

STEP 1 **First,** a translation maps P to P'.

STEP 2 **Then,** a reflection in a line k parallel to the direction of the translation maps P' to P''.



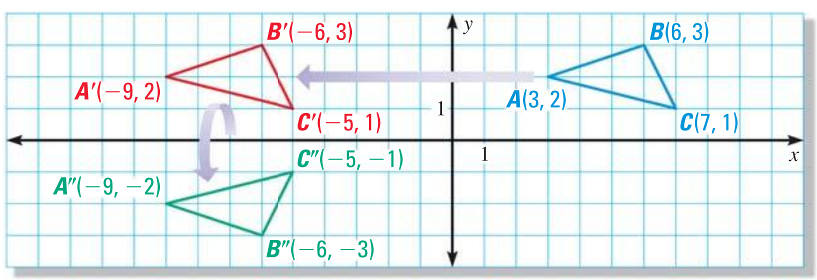
EXAMPLE 1 Find the image of a glide reflection

The vertices of $\triangle ABC$ are $A(3, 2)$, $B(6, 3)$, and $C(7, 1)$. Find the image of $\triangle ABC$ after the glide reflection.

Translation: $(x, y) \rightarrow (x - 12, y)$
Reflection: in the x-axis

Solution

Begin by graphing $\triangle ABC$. Then graph $\triangle A'B'C'$ after a translation 12 units left. Finally, graph $\triangle A''B''C''$ after a reflection in the x-axis.



AVOID ERRORS
The line of reflection must be parallel to the direction of the translation to be a glide reflection.

✓ GUIDED PRACTICE for Example 1

1. Suppose $\triangle ABC$ in Example 1 is translated 4 units down, then reflected in the y-axis. What are the coordinates of the vertices of the image?
2. In Example 1, *describe* a glide reflection from $\triangle A''B''C''$ to $\triangle ABC$.

COMPOSITIONS When two or more transformations are combined to form a single transformation, the result is a **composition of transformations**. A glide reflection is an example of a composition of transformations.

In this lesson, a composition of transformations uses isometries, so the final image is congruent to the preimage. This suggests the Composition Theorem.

THEOREM *For Your Notebook*

THEOREM 9.4 Composition Theorem

The composition of two (or more) isometries is an isometry.

Proof: Exs. 35–36, p. 614

EXAMPLE 2 Find the image of a composition

The endpoints of $\overline{RS}$ are $R(1, -3)$ and $S(2, -6)$. Graph the image of $\overline{RS}$ after the composition.

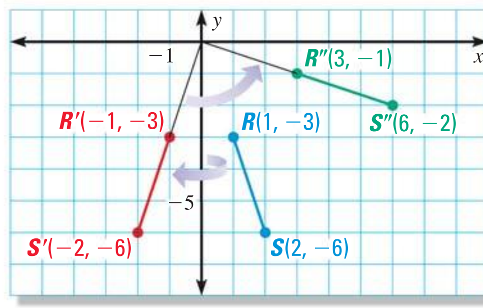
Reflection: in the y-axis
Rotation: 90° about the origin

Solution

STEP 1 **Graph** $\overline{RS}$.

STEP 2 **Reflect** $\overline{RS}$ in the y-axis. $\overline{R'S'}$ has endpoints $R'(-1, -3)$ and $S'(-2, -6)$.

STEP 3 **Rotate** $\overline{R'S'}$ 90° about the origin. $\overline{R''S''}$ has endpoints $R''(3, -1)$ and $S''(6, -2)$.



AVOID ERRORS
Unless you are told otherwise, do the transformations in the order given.

TWO REFLECTIONS Compositions of two reflections result in either a translation or a rotation, as described in Theorems 9.5 and 9.6.

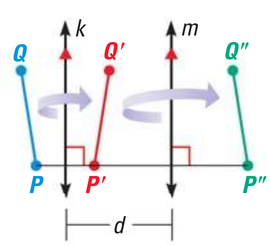
THEOREM *For Your Notebook*

THEOREM 9.5 Reflections in Parallel Lines Theorem

If lines k and m are parallel, then a reflection in line k followed by a reflection in line m is the same as a translation.

If P'' is the image of P, then:

1. $\overline{PP''}$ is perpendicular to k and m, and
2. $PP'' = 2d$, where d is the distance between k and m.



Proof: Ex. 37, p. 614

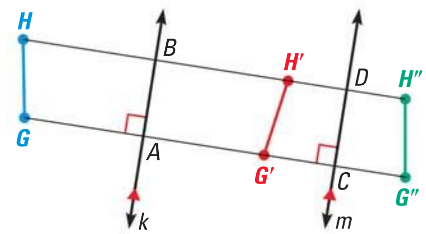
EXAMPLE 3 Use Theorem 9.5

In the diagram, a reflection in line k maps $\overline{GH}$ to $\overline{G'H'}$. A reflection in line m maps $\overline{G'H'}$ to $\overline{G''H''}$. Also, $HB = 9$ and $DH'' = 4$.

a. Name any segments congruent to each segment: $\overline{HG}$, $\overline{HB}$, and $\overline{GA}$.

b. Does $AC = BD$? Explain.

c. What is the length of $\overline{GG''}$?



Solution

a. $\overline{HG} \cong \overline{H'G'}$, and $\overline{HG} \cong \overline{H''G''}$. $\overline{HB} \cong \overline{H'B}$. $\overline{GA} \cong \overline{G'A}$.

b. Yes, $AC = BD$ because $\overline{GG''}$ and $\overline{HH''}$ are perpendicular to both k and m, so $\overline{BD}$ and $\overline{AC}$ are opposite sides of a rectangle.

c. By the properties of reflections, $H'B = 9$ and $H'D = 4$. Theorem 9.5 implies that $GG'' = HH'' = 2 \cdot BD$, so the length of $\overline{GG''}$ is $2(9 + 4)$, or 26 units.

✓ GUIDED PRACTICE for Examples 2 and 3

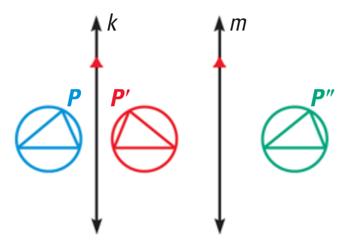
3. Graph $\overline{RS}$ from Example 2. Do the rotation first, followed by the reflection. Does the order of the transformations matter? *Explain.*

4. In Example 3, part (c), *explain* how you know that $GG'' = HH''$.

Use the figure below for Exercises 5 and 6. The distance between line k and line m is 1.6 centimeters.

5. The preimage is reflected in line k, then in line m. *Describe* a single transformation that maps the blue figure to the green figure.

6. What is the distance between P and P''? If you draw $\overline{PP'}$, what is its relationship with line k? *Explain.*

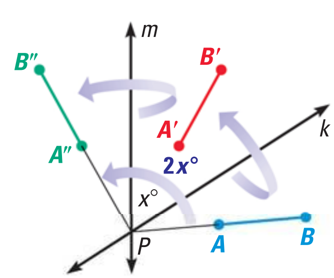


THEOREM — *For Your Notebook*

THEOREM 9.6 Reflections in Intersecting Lines Theorem

If lines k and m intersect at point P, then a reflection in k followed by a reflection in m is the same as a rotation about point P.

The angle of rotation is $2x°$, where $x°$ is the measure of the acute or right angle formed by k and m.

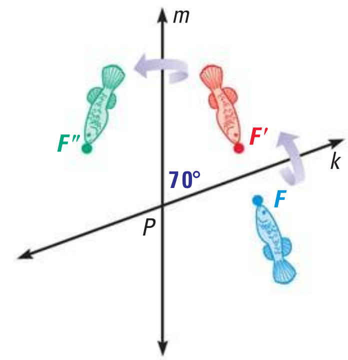


$m\angle BPB'' = 2x°$

Proof: Ex. 38, p. 614

EXAMPLE 4 Use Theorem 9.6

In the diagram, the figure is reflected in line *k*. The image is then reflected in line *m*. Describe a single transformation that maps *F* to *F″*.



Solution

The measure of the acute angle formed between lines *k* and *m* is 70°. So, by Theorem 9.6, a single transformation that maps *F* to *F″* is a 140° rotation about point *P*.

You can check that this is correct by tracing lines *k* and *m* and point *F*, then rotating the point 140°.

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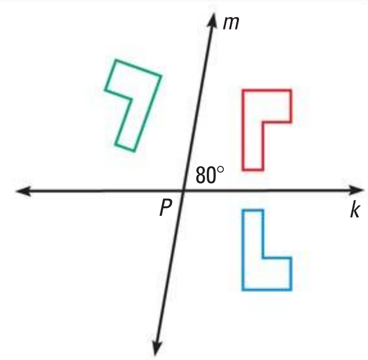
✓ GUIDED PRACTICE for Example 4



7. In the diagram at the right, the preimage is reflected in line *k*, then in line *m*. *Describe* a single transformation that maps the blue figure onto the green figure.

8. A rotation of 76° maps *C* to *C′*. To map *C* to *C′* using two reflections, what is the angle formed by the intersecting lines of reflection?

9.5 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 7, 17, and 27

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 25, 29, and 34

SKILL PRACTICE

1. **VOCABULARY** Copy and complete: In a glide reflection, the direction of the translation must be __?__ to the line of reflection.

2. ★ **WRITING** *Explain* why a glide reflection is an isometry.

EXAMPLE 1 on p. 608 for Exs. 3–6

GLIDE REFLECTION **The endpoints of $\overline{CD}$ are $C(2, -5)$ and $D(4, 0)$. Graph the image of $\overline{CD}$ after the glide reflection.**

3. **Translation:** $(x, y) \rightarrow (x, y - 1)$
   Reflection: in the y-axis

4. **Translation:** $(x, y) \rightarrow (x - 3, y)$
   Reflection: in $y = -1$

5. **Translation:** $(x, y) \rightarrow (x, y + 4)$
   Reflection: in $x = 3$

6. **Translation:** $(x, y) \rightarrow (x + 2, y + 2)$
   Reflection: in $y = x$

EXAMPLE 2
on p. 609
for Exs. 7–14

GRAPHING COMPOSITIONS **The vertices of $\triangle PQR$ are $P(2, 4)$, $Q(6, 0)$, and $R(7, 2)$. Graph the image of $\triangle PQR$ after a composition of the transformations in the order they are listed.**

7. **Translation:** $(x, y) \to (x, y - 5)$
   Reflection: in the y-axis

8. **Translation:** $(x, y) \to (x - 3, y + 2)$
   Rotation: 90° about the origin

9. **Translation:** $(x, y) \to (x + 12, y + 4)$
   Translation: $(x, y) \to (x - 5, y - 9)$

10. **Reflection:** in the x-axis
    Rotation: 90° about the origin

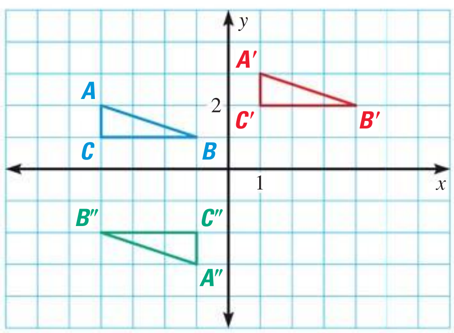
REVERSING ORDERS **Graph $\overline{F''G''}$ after a composition of the transformations in the order they are listed. Then perform the transformations in reverse order. Does the order affect the final image $\overline{F''G''}$?**

11. $F(-5, 2)$, $G(-2, 4)$
    Translation: $(x, y) \to (x + 3, y - 8)$
    Reflection: in the x-axis

12. $F(-1, -8)$, $G(-6, -3)$
    Reflection: in the line $y = 2$
    Rotation: 90° about the origin

DESCRIBING COMPOSITIONS ***Describe*** **the composition of transformations.**

13.



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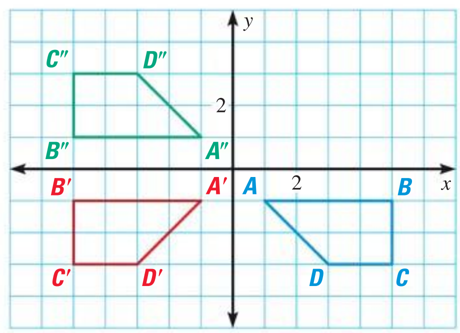
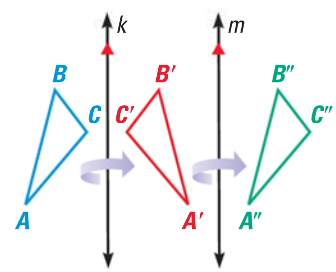


EXAMPLE 3
on p. 610
for Exs. 15–19

USING THEOREM 9.5 **In the diagram, $k \parallel m$, $\triangle ABC$ is reflected in line k, and $\triangle A'B'C'$ is reflected in line m.**

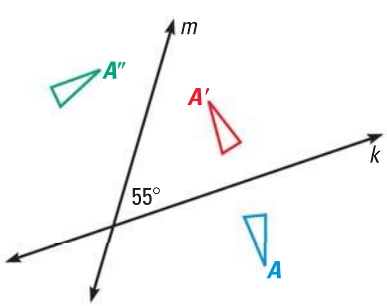


15. A translation maps $\triangle ABC$ onto which triangle?
16. Which lines are perpendicular to $\overleftrightarrow{AA''}$?
17. Name two segments parallel to $\overleftrightarrow{BB''}$.
18. If the distance between k and m is 2.6 inches, what is the length of $\overleftrightarrow{CC''}$?
19. Is the distance from B' to m the same as the distance from B'' to m? *Explain.*

EXAMPLE 4
on p. 611
for Exs. 20–21

USING THEOREM 9.6 **Find the angle of rotation that maps A onto A''.**

20.



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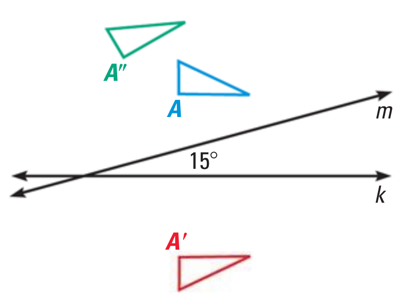
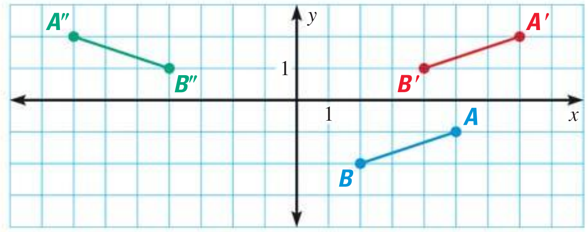


22. **ERROR ANALYSIS** A student described the translation of $\overline{AB}$ to $\overline{A'B'}$ followed by the reflection of $\overline{A'B'}$ to $\overline{A''B''}$ in the y-axis as a glide reflection. *Describe* and correct the student's error.



USING MATRICES **The vertices of $\triangle PQR$ are $P(1, 4)$, $Q(3, -2)$, and $R(7, 1)$. Use matrix operations to find the image matrix that represents the composition of the given transformations. Then graph $\triangle PQR$ and its image.**

23. **Translation:** $(x, y) \rightarrow (x, y + 5)$
Reflection: in the y-axis

24. **Reflection:** in the x-axis
Translation: $(x, y) \rightarrow (x - 9, y - 4)$

25. ★ **OPEN-ENDED MATH** Sketch a polygon. Apply three transformations of your choice on the polygon. What can you say about the congruence of the preimage and final image after multiple transformations? *Explain*.

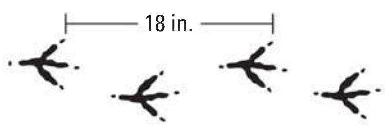
26. **CHALLENGE** The vertices of $\triangle JKL$ are $J(1, -3)$, $K(2, 2)$, and $L(3, 0)$. Find the image of the triangle after a $180°$ rotation about the point $(-2, 2)$, followed by a reflection in the line $y = -x$.

PROBLEM SOLVING

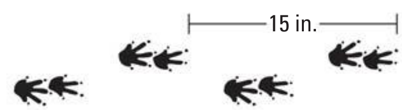
EXAMPLE 1
on p. 608
for Exs. 27–30

ANIMAL TRACKS **The left and right prints in the set of animal tracks can be related by a glide reflection. Copy the tracks and *describe* a translation and reflection that combine to create the glide reflection.**

27. bald eagle (2 legs)



28. armadillo (4 legs)



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29. ★ **MULTIPLE CHOICE** Which is *not* a glide reflection?

Ⓐ The teeth of a closed zipper

Ⓑ The tracks of a walking duck

Ⓒ The keys on a computer keyboard

Ⓓ The red squares on two adjacent rows of a checkerboard

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30. **ROWING** *Describe* the transformations that are combined to represent an eight-person rowing shell.

SWEATER PATTERNS In Exercises 31–33, *describe* the transformations that are combined to make each sweater pattern.

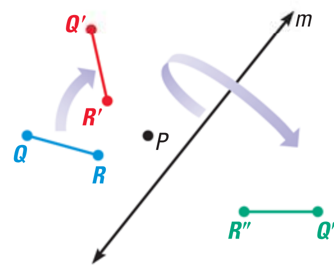
31. 

32. 

33. 

34. ★ **SHORT RESPONSE** Use Theorem 9.5 to *explain* how you can make a glide reflection using three reflections. How are the lines of reflection related?

35. **PROVING THEOREM 9.4** Write a plan for proof for one case of the Composition Theorem.

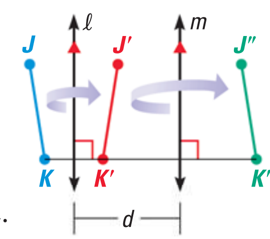


GIVEN ▶ A rotation about P maps Q to Q' and R to R'. A reflection in m maps Q' to Q'' and R' to R''.

PROVE ▶ $QR = Q''R''$

36. **PROVING THEOREM 9.4** A composition of a rotation and a reflection, as in Exercise 35, is one case of the Composition Theorem. List all possible cases, and prove the theorem for another pair of compositions.

37. **PROVING THEOREM 9.5** Prove the Reflection in Parallel Lines Theorem.

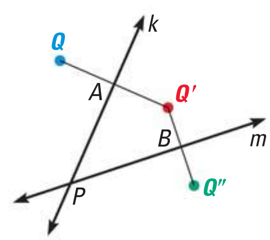


GIVEN ▶ A reflection in line ℓ maps $\overline{JK}$ to $\overline{J'K'}$, a reflection in line m maps $\overline{J'K'}$ to $\overline{J''K''}$, and $\ell \parallel m$.

PROVE ▶ **a.** $\overleftrightarrow{KK''}$ is perpendicular to ℓ and m.

b. $KK'' = 2d$, where d is the distance between ℓ and m.

38. **PROVING THEOREM 9.6** Prove the Reflection in Intersecting Lines Theorem.



GIVEN ▶ Lines k and m intersect at point P. Q is any point not on k or m.

PROVE ▶ **a.** If you reflect point Q in k, and then reflect its image Q' in m, Q'' is the image of Q after a rotation about point P.

b. $m\angle QPQ'' = 2(m\angle APB)$

Plan for Proof First show $k \perp \overline{QQ'}$ and $\overline{QA} \cong \overline{Q'A}$. Then show $\triangle QAP \cong \triangle Q'AP$. In the same way, show $\triangle Q'BP \cong \triangle Q''BP$. Use congruent triangles and substitution to show that $\overline{QP} \cong \overline{Q''P}$. That proves part (a) by the definition of a rotation. Then use congruent triangles to prove part (b).

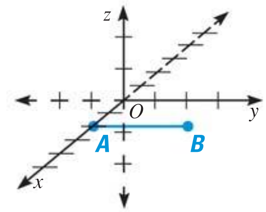
39. **VISUAL REASONING** You are riding a bicycle along a flat street.

a. What two transformations does the wheel's motion use?

b. *Explain* why this is not a composition of transformations.

★ = STANDARDIZED TEST PRACTICE

40. **MULTI-STEP PROBLEM** A point in space has three coordinates (x, y, z). From the origin, a point can be forward or back on the x-axis, left or right on the y-axis, and up or down on the z-axis. The endpoints of segment $\overline{AB}$ in space are $A(2, 0, 0)$ and $B(2, 3, 0)$, as shown at the right.



a. Rotate $\overline{AB}$ 90° about the x-axis with center of rotation A. What are the coordinates of $\overline{A'B'}$?

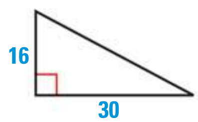
b. Translate $\overline{A'B'}$ using the vector $\langle 4, 0, -1\rangle$. What are the coordinates of $\overline{A''B''}$?

41. **CHALLENGE** *Justify* the following conjecture or provide a counterexample.

Conjecture When performing a composition of two transformations of the *same type*, order does not matter.

MIXED REVIEW

Find the unknown side length. Write your answer in simplest radical form. *(p. 433)*

42.


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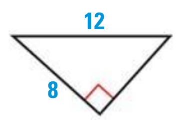

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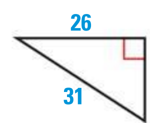

PREVIEW
Prepare for Lesson 9.6 in Exs. 45–48.

The coordinates of $\triangle PQR$ are $P(3, 1)$, $Q(3, 3)$, and $R(6, 1)$. Graph the image of the triangle after the translation. *(p. 572)*

45. $(x, y) \to (x + 3, y)$

46. $(x, y) \to (x - 3, y)$

47. $(x, y) \to (x, y + 2)$

48. $(x, y) \to (x + 3, y + 2)$

QUIZ *for Lessons 9.3–9.5*

The vertices of $\triangle ABC$ are $A(7, 1)$, $B(3, 5)$, and $C(10, 7)$. Graph the reflection in the line. *(p. 589)*

1. y-axis

2. $x = -4$

3. $y = -x$

Find the coordinates of the image of $P(2, -3)$ after the rotation about the origin. *(p. 598)*

4. 180° rotation

5. 90° rotation

6. 270° rotation

The vertices of $\triangle PQR$ are $P(-8, 8)$, $Q(-5, 0)$, and $R(-1, 3)$. Graph the image of $\triangle PQR$ after a composition of the transformations in the order they are listed. *(p. 608)*

7. **Translation:** $(x, y) \to (x + 6, y)$
Reflection: in the y-axis

8. **Reflection:** in the line $y = -2$
Rotation: 90° about the origin

9. **Translation:** $(x, y) \to (x - 5, y)$
Translation: $(x, y) \to (x + 2, y + 7)$

10. **Rotation:** 180° about the origin
Translation: $(x, y) \to (x + 4, y - 3)$